

EE 477
Digital Signal Processing

8b

IIR Systems

Inverse z-Transform

- Since $Y(z) = H(z) X(z)$, we can use the system function $H(z)$ and the input $X(z)$ to find the output transform $Y(z)$
- If we have $Y(z)$, then how should we get $y[n]$??
- Usual way: manipulate $Y(z)$ into a sum of recognizable terms from a transform table

Inverse Transform Tables

- Essential transform pairs:

$$x[n - n_0] \Leftrightarrow z^{-n_0} X(z)$$

$$\delta[n] \Leftrightarrow 1$$

$$\delta[n - n_0] \Leftrightarrow z^{-n_0}$$

$$u[n] \Leftrightarrow \frac{1}{1 - z^{-1}}$$

$$a^n u[n] \Leftrightarrow \frac{1}{1 - az^{-1}}$$

Partial Fraction Expansion

- Separate ratio of polynomials into a sum of simpler polynomial ratios, then “take the inverse transform” by matching to the table

- Ex:

$$\frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1})(1 - z^{-1})} = \frac{A}{1 - a_1 z^{-1}} + \frac{B}{1 - z^{-1}}$$

$Aa_1^n u[n]$ $B u[n]$

$$= \frac{A + B - (A + Ba_1)z^{-1}}{(1 - a_1 z^{-1})(1 - z^{-1})}$$

Partial Fraction (cont.)

- Equate A and B terms with numerator

$$A + B = b_0$$

$$A + Ba_1 = -b_1$$

- Solving:

$$B = \frac{b_0 + b_1}{1 - a_1}$$

$$A = \frac{b_0 a_1 + b_1}{a_1 - 1}$$

Partial Fraction (cont.)

- Alternative: evaluate by isolating the coefficients

$$H(z) = \frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1})(1 - z^{-1})} = \frac{A}{1 - a_1 z^{-1}} + \frac{B}{1 - z^{-1}}$$

$$H(z)(1 - a_1 z^{-1}) = \frac{b_0 + b_1 z^{-1}}{(1 - z^{-1})} = A + \frac{B(1 - a_1 z^{-1})}{1 - z^{-1}}$$

\swarrow $0 (z=a_1)$

$$H(z)(1 - a_1 z^{-1}) \Big|_{z=a_1} = \frac{b_0 + b_1 a_1^{-1}}{(1 - a_1^{-1})} = A$$

Partial Fraction (cont.)

- This approach works fine for distinct poles. If duplicate poles, need to use a similar systematic procedure (see an advanced DSP text, or take EE577)

Second-order System

- Typically implement *higher-order* IIR systems using *second-order* blocks
- Why?
 - Helps isolate coefficients to lower quantization sensitivity
 - Each block implements a pair of complex conjugate poles (or real)

Second-order Response

- General second order:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$
$$= -\frac{b_2}{a_2} + \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}}$$

- Unit sample response form:

$$h[n] = -\left(\frac{b_2}{a_2}\right) \delta[n] + A_1 (p_1)^n u[n] + A_2 (p_2)^n u[n]$$

2nd-order poles: real

- The two poles of the 2nd-order system are either both real, or are complex conjugates
- If p_1 and p_2 are real, impulse response is two decaying exponentials of the form p_1^n and p_2^n (e.g., $\left(\frac{1}{3}\right)^n u[n]$ or $\left(-\frac{1}{8}\right)^n u[n]$)

2nd-order poles: complex

- If p_1 and p_2 are complex conjugates, express in polar form as $p_1 = re^{j\theta}$

$$p_2 = re^{-j\theta} = p_1^*$$

$$\begin{aligned} h[n] &= -\left(\frac{b_2}{a_1}\right)\delta[n] + A_1(re^{j\theta})^n u[n] + A_2(re^{-j\theta})^n u[n] \\ &= -\left(\frac{b_2}{a_1}\right)\delta[n] + A_1 r^n e^{j\theta n} u[n] + A_2 r^n e^{-j\theta n} u[n] \end{aligned}$$

Complex poles (cont.)

- Denominator:

$$\begin{aligned}
 1 - a_1 z^{-1} - a_2 z^{-2} &= (1 - p_1 z^{-1})(1 - p_2 z^{-1}) \\
 &= (1 - r e^{j\theta} z^{-1})(1 - r e^{-j\theta} z^{-1}) \\
 &= 1 - 2r \cos\theta z^{-1} + r^2 z^{-2} \\
 &= 1 - \underbrace{2r \cos\theta}_{a_1} z^{-1} - \underbrace{\{-r^2\}}_{a_2} z^{-2}
 \end{aligned}$$

- Note influence of pole locations on coefficients a_1 and a_2